ČECH-COMPLETENESS AND RELATED PROPERTIES OF THE GENERALIZED COMPACT-OPEN TOPOLOGY

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Abstract. The generalized compact-open topology τ_C on partial continuous functions with closed domains in X and values in Y is studied. If Y is a non-countably compact Čech-complete space with a G_δ-diagonal, then τ_C is Čech-complete, sieve complete and satisfies the p-space property of Arhangel'skiĭ, respectively, if and only if X is Lindelöf and locally compact. Lindelöfness, paracompactness and normality of τ_C is also investigated. New results are obtained on Čech-completeness, sieve completeness and the p-space property for the compact-open topology on the space of continuous functions with a general range Y.

1. Introduction

The *generalized compact-open topology* τ_C on the space of partial continuous functions with closed domains was introduced by *J. Back* in [Ba] in connection with investigating utility functions emerging in mathematical economics. It proved to be a useful tool in studying convergence of dynamic programming models [Wh], [La], as well as in applications to the theory of differential equations [BC]. This new interest in τ_C complements the attention paid to spaces of partial maps in the past [Za], [Ku1], [Ku2], [AA], [AB], [BB], [Se], and more recently in [Fi], [St], [KS], [BCH1], [BCH2], [DN1], [DN2], [HZ1], [HZ2].

Various topological properties of τ_C have already been established, e.g. separation axioms in [Ho1], complete metrizability in [Ho2] and other completeness type properties in [HZ1] and [NZ], respectively; also, in [DN1], [DN2] the authors study topological properties of spaces of partial maps in a more general setting.

It is the purpose of this paper to continue in this research by investigating Čech-completeness of τ_C. In the process, sieve completeness and the p-space property [Arh] of τ_C is proved to be equivalent to its Čech-completeness in the most interesting cases, which is much like the situation with the compact-open topology τ_{CO} on the space of continuous functions (see [MN1, MN2]). It is worth noticing however, that despite the close connection between τ_C and τ_{CO}, properties of these topologies do not always coincide. On the contrary, in some cases (like Baireness, or weak α-favorability [HZ1]), the generalized compact-open topology exhibits properties resembling those of the Fell hyperspace topology τ_F, which makes τ_C a true mixture of τ_{CO} and τ_F. In Section 2 we collect some definitions and auxiliary results for the topologies we are to study. In Section 3, we prove the main results of the

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paper: a characterization of Čech-completeness, sieve completeness and the p-space property, respectively. Extending a theorem of *Holá* [Ho2], a full characterization of complete metrizability of τ_C is also given. In Section 4, we obtain sufficient conditions for Čech-completeness, sieve completeness and the p-space property, respectively, of the compact-open topology $(C(X,Y),\tau_{CO})$ for Y with a G_δ-diagonal, thus generalizing results of *McCoy* and *Ntantu* [MN1, MN2]. In Section 5, the relationship between Lindelöfness, paracompactness and normality of τ_C is explored. Surprisingly, normality of τ_C implies its Čech-completeness and the reverse implication is also true under some restrictions. An application of paracompactness of τ_C to continuous extensions of partial functions is given.

2. Preliminaries

Unless otherwise noted, all spaces are nontrivial Hausdorff spaces. If X is a topological space, then B^c, $\text{int}B$, and $\overline{B}$ (or $\overline{B}^X$) will stand for the complement, interior, and closure of $B \subseteq X$, respectively. Denote by $CL(X)$ the family of nonempty closed subsets of X, and by $K(X)$ the nonempty compact subsets of X. For any $B \in CL(X)$, and a topological space Y, $C(B,Y)$ will stand for the space of continuous functions from B to Y (so-called *partial maps*). Denote by

$$\mathcal{P} = \mathcal{P}(X,Y) = \bigcup\{C(B,Y) : B \in CL(X)\}$$

the family of all partial maps. We will identify a partial map f with its graph $\Gamma(f) \in CL(X \times Y)$. If Y is a Tychonoff space and cY is a fixed compactification of Y, then $\mathcal{P} \subseteq CL(X \times cY)$, since if $(x_\lambda, f(x_\lambda))$ is a net in $\Gamma(f) \in \mathcal{P}$ converging to some $(x,y) \in X \times cY$, then $x_\lambda \to x$ and $f(x_\lambda) \to y$; further, by continuity of f, $f(x_\lambda) \to f(x)$, so $y = f(x)$ and $(x,y) \in \Gamma(f)$.

Define the so-called *generalized compact-open topology* τ_C on $\mathcal{P}$ as the topology having subbase elements of the form

$$[U] = \{f \in \mathcal{P} : (dom f) \cap U \neq \emptyset\},$$
$$[K : I] = \{f \in \mathcal{P} : f(K \cap (dom f)) \subseteq I\},$$

where U is open in X, $K \in K(X)$ and I is an open (possibly empty) subset of Y. We can assume that the I's are members of some fixed open base for Y, or empty.

The *compact-open topology* [En, MN1] τ_{CO} on $C(X,Y)$ has subbase elements of the form $\{f \in C(X,Y) : f(K) \subseteq I\}$, where $K \in K(X)$ and $I \subseteq Y$ is open.

Denote by τ_F the so-called *Fell topology* [Be, KT] on $CL(X)$ having subbase elements of the form

$$V^- = \{A \in CL(X) : A \cap V \neq \emptyset\}$$

with V open in X, plus sets of the form

$$(K^c)^+ = \{A \in CL(X) : A \subseteq K^c\},$$

with $K \in K(X)$. It was observed by *Fell* that if X is locally compact, then $(CL(X) \cup \{\emptyset\}, \tau_F)$ is a compact Hausdorff space [Fe]; furthermore, since $CL(X) = X^-$ is an open subspace of $(CL(X) \cup \{\emptyset\}, \tau_F)$, $(CL(X), \tau_F)$ is locally compact, if X is. We will use τ_F to denote the Fell topology on $CL(X \times cY)$ as well as on $\mathcal{P}$. If $L \in CL(X \times Y)$, denote

$$L[x] = \{y \in Y : (x,y) \in L\}$$

(i.e. L can be viewed as a multifunction with a closed graph [HM]). Let p_X be the projection map from $X \times Y$ onto X, and ω the non-negative integers. For notions not defined in the paper see [En].

In the following two propositions we explore the relationship between the various topologies defined in this section:

Proposition 2.1.

(i) *X and $(CL(X), \tau_F)$ embed in $(\mathcal{P}, \tau_C)$; further, they embed as closed subsets, if X is locally compact.*

(ii) *Y and $(C(X,Y), \tau_{CO})$ embed as closed subsets in $(\mathcal{P}, \tau_C)$.*

Proof. (i) $x \mapsto \{x\}$ is a closed embedding of X into $(CL(X), \tau_F)$. Let $y \in Y$ be fixed, and for each $A \in CL(X)$ define $f_A \in \mathcal{P}$ via $f_A(x) = y$ for all $x \in A$, then $\phi : A \mapsto f_A$ is an embedding of $(CL(X), \tau_F)$ into $(\mathcal{P}, \tau_C)$. Let $f \in \mathcal{P}$ be such that it has at least two distinct values $y_1 \neq y_2$, and let I_1, I_2 be Y-open disjoint neighborhoods of y_1, y_2, respectively. If X is locally compact, we can find $K_1, K_2 \in K(X)$ such that $f(B \cap K_1) \subseteq I_1$, and $f(B \cap K_2) \subseteq I_2$, where $B = dom f$. Then

$$f \in [K_1 : I_1] \cap [K_2 : I_2] \subseteq \mathcal{P} \setminus \phi(CL(X)),$$

so $\phi(CL(X))$ is closed in $(\mathcal{P}, \tau_C)$.

(ii) Assigning to each $y \in Y$ the function $f_y \in C(X,Y)$, defined via $f_y(x) = y$ for all $x \in X$, sets up a closed embedding of Y into $(C(X,Y), \tau_{CO})$. Moreover, the identity map $i : (C(X,Y), \tau_{CO}) \to (\mathcal{P}, \tau_C)$ is clearly an embedding, and if $f \in \mathcal{P} \setminus i(C(X,Y))$ then there is $x \notin dom f$, so

$$f \in [\{x\} : \emptyset] \subseteq \mathcal{P} \setminus i(C(X,Y));$$

thus, $i(C(X,Y))$ is closed in $(\mathcal{P}, \tau_C)$. □

Proposition 2.2. *If X is locally compact and Y is Tychonoff, then $(\mathcal{P}, \tau_C)$ is a subspace of $(CL(X \times cY), \tau_F)$.*

Proof. To prove that $\tau_C \subseteq \tau_F$ on $\mathcal{P}$, let U be an open set in X and consider the τ_C-open set $[U]$. If $f \in [U]$, then $(U \times cY)^- \cap \mathcal{P}$ is a τ_F-neighborhood of f in $\mathcal{P}$ contained in $[U]$. Now, consider a τ_C-open set $[K : I]$, where $K \in K(X)$ and I is open in Y, and choose a cY-open set J with $I = J \cap Y$. It is easy to verify that

$$[K : I] = ((K \times (cY \setminus J))^c)^+ \cap \mathcal{P}.$$

Conversely, let $U \subseteq X$ and $V \subseteq cY$ be open, and $f \in (U \times V)^- \cap \mathcal{P}$. Then $(x, f(x)) \in U \times (V \cap Y)$ for some $x \in B := dom f$, so local compactness of X and continuity of f imply that there is an X-open neighborhood O_x of x with compact closure such that $\overline{O}_x \subseteq U$ and $f(\overline{O}_x \cap B) \subseteq V \cap Y$. Then

$$f \in [O_x] \cap [\overline{O}_x : V \cap Y] \subseteq (U \times V)^- \cap \mathcal{P}.$$

Finally, let K be a compact set in $X \times cY$ and $f \in (K^c)^+$. By local compactness, every compact set in $X \times cY$ missing f is contained in a finite union of product sets with compact factors missing f, so, without loss of generality, assume $K = K_1 \times K_2$, where $K_1 \in K(X)$, $K_2 \in K(cY)$. It is easy to verify that $\Gamma(f) \subseteq (K_1 \times K_2)^c$ if and only if $f(K_1 \cap (dom f)) \subseteq K_2^c$. In other words,

$$[(K_1 \times K_2)^c]^+ \cap \mathcal{P} = [K_1 : (K_2^c) \cap Y].$$

This implies, that $(K^c)^+ \cap \mathcal{P} \in \tau_C$, so $\tau_F \restriction_{\mathcal{P}} \subseteq \tau_C$ indeed. □

Remark 2.3. Observe, that for X locally compact and any Y, we can use the above argument to show that the Fell topology from $CL(X \times Y)$ on $\mathcal{P}$ is weaker than τ_C. However, the converse is not true in general: indeed, consider $X = \{0\} \cup \{\frac{1}{n} : n \in \omega\}$, $Y = \omega$, both with the natural topology, and define $f_n : \{0, \frac{1}{n}\} \to Y$ via $f_n(0) = 0$, $f_n(\frac{1}{n}) = n$ for each $n \in \omega$, and $f : \{0\} \to Y$ via $f(0) = 0$. Then the sequence $\{f_n\}_n$ τ_F-converges to f, but fails to τ_C-converge to f, since $f \in [X : \{0\}]$ but $f_n \notin [X : \{0\}]$ for all n. □

It is not hard to show that $(\mathcal{P}, \tau_C)$ is always T_1; however, higher separation axioms impose some restrictions on X and Y, as was shown in [Ho1, Ho2] for Tychonoff X, Y. We have the following more refined result on separation axioms with a new proof:

Proposition 2.4. *The following are equivalent:*

(i) *$(\mathcal{P}, \tau_C)$ is Hausdorff (Tychonoff);*
(ii) *X is locally compact and Y is Hausdorff (Tychonoff).*

Proof. (i)⇒(ii) Y has the relevant properties by Proposition 2.1(ii). Proposition 2.1(i) implies, that $(CL(X), \tau_F)$ is Hausdorff, so X is locally compact by [Be, Proposition 5.1.2]. We can also give a direct proof of local compactness of X: let $x \in X$ and $y_1 \neq y_2$ be two different points of Y. Then $\{(x, y_1)\}$ and $\{(x, y_2)\}$ are two different elements of $\mathcal{P}$, so there are disjoint neighborhoods H_1, H_2 of $\{(x, y_1)\}$ and $\{(x, y_2)\}$, respectively in $(\mathcal{P}, \tau_C)$. We can assume that

$$H_1 = [U_1] \cap [K_1 : I_1] \cap [L_1 : \emptyset] \text{ and } H_2 = [U_2] \cap [K_2 : I_2] \cap [L_2 : \emptyset],$$

where U_1, U_2 are nonempty X-open sets, I_1, I_2 are Y-open, $K_1, K_2, L_1, L_2 \in K(X)$, $U_1 \subseteq L_1^c$, and $U_2 \subseteq L_2^c$. Since $\{(x, y_1)\} \in H_1$ and $\{(x, y_2)\} \in H_2$, we have $x \in U_1 \cap U_2$. Moreover, $U_1 \cap U_2 \subseteq K_1 \cup K_2$, since otherwise, if $z \in U_1 \cap U_2 \setminus K_1 \cup K_2$, then $\{(z, y_1)\} \in H_1 \cap H_2$, a contradiction.

(ii)⇒(i) Hausdorffness: let $f_0, f_1 \in \mathcal{P}$ be distinct. If $dom f_0 \neq dom f_1$, without loss of generality, take some $x \in dom f_0 \setminus dom f_1$. Then there is an X-open U and $K \in K(X)$ with $x \in U \subseteq K \subseteq X \setminus dom f_1$. It follows that $[U]$ and $[K : \emptyset]$ are disjoint τ_C-open neighborhoods of f_0 and f_1, respectively. On the other hand, if $B = dom f_0 = dom f_1$, then $f_0(x) \neq f_1(x)$ for some $x \in B$, so choosing disjoint Y-open neighborhoods I_0, I_1 of $f_0(x), f_1(x)$, respectively, as well as an X-open neighborhood U of x with compact closure such that $f_i(\overline{U} \cap B) \subseteq I_i$ for $i = 0, 1$, we have that $[U] \cap [\overline{U} : I_0]$ and $[U] \cap [\overline{U} : I_1]$ are disjoint τ_C-open neighborhoods of f_0 and f_1, respectively.

Tychonoffness: $X \times \beta Y$ is Hausdorff and locally compact (βY is the Čech-Stone compactification of Y), so $(CL(X \times \beta Y), \tau_F)$ is Tychonoff [Be, Proposition 5.1.2], and, by our Proposition 2.2, so is $(\mathcal{P}, \tau_C)$. □

3. Čech-completeness and related properties of $(\mathcal{P}, \tau_C)$

From now on, cY is a fixed Hausdorff compactification of a Tychonoff space Y. Recall, that Y is *Čech-complete* [En], if Y is G_δ in its compactification cY. Also, Y has a *G_δ-diagonal*, if $\{(y, y) : y \in Y\}$ is a G_δ-set in $Y \times Y$, equivalently [Gr], if there exists a sequence $\{\mathcal{V}_m\}_m$ of cY-open covers of Y such that for each $y \in Y$, $\{y\} = Y \cap \bigcap_m \mathrm{St}(y, \mathcal{V}_m)$, where

$$\mathrm{St}(y, \mathcal{V}_m) = \bigcup \{V \in \mathcal{V}_m : y \in V\}.$$

We will say that Y is a *p-space* [Arh, Gr], provided there is a *feathering* for Y, i.e. there is a sequence $\{\mathcal{V}_m\}_m$ of open covers of Y in cY such that $\bigcap_m \mathrm{St}(y, \mathcal{V}_m) \subseteq Y$ for all $y \in Y$. Analogously, we can define *cp-spaces*, if we require $\bigcap_m \mathrm{St}(K, \mathcal{V}_m) \subseteq Y$ for all $K \in K(Y)$, where

$$\mathrm{St}(K, \mathcal{V}_m) = \bigcup\{V \in \mathcal{V}_m : K \cap V \neq \emptyset\}.$$

It is easy to see that a Čech-complete space is a *cp*-space, which in turn is a *p*-space. On the other hand, a paracompact *p*-space is a *cp*-space: indeed, Y is a paracompact *p*-space provided there is a metric space Z and a perfect map $f : Y \to Z$ [Gr]. For each $m \geq 1$ consider the Y-cover

$$\mathcal{V}'_m = \{f^{-1}(U) : U \text{ open } Z\text{-ball of radius } \frac{1}{m}\}$$

and define $\mathcal{V}_m = \{V \subseteq cY \text{ open} : Y \cap V \in \mathcal{V}'_m\}$. Take some $K \in K(Y)$, and assume there exists $p \in (cY \setminus Y) \cap \bigcap_m \mathrm{St}(K, \mathcal{V}_m)$. Then for each m we can find $V_m \in \mathcal{V}_m$ with $p \in V_m$ and $K \cap V_m \neq \emptyset$; let $y_m \in K \cap V_m$. The sequence $\{y_m\}_m$ has a cluster point $y \in K$, and by continuity of f, $f(y)$ is a cluster point of $\{f(y_m)\}_m$. The set $L = f^{-1}(f(y))$ is compact in Y and hence in cY as well, thus, there exists a cY-open W containing p such that $\overline{W}^{cY} \subseteq cY \setminus L$. Since f is closed, $f(Y \cap \overline{W}^{cY})$ is a closed set missing $f(y)$. Note that each $f(Y \cap V_m)$ is a Z-open $\frac{1}{m}$-ball containing $f(y_m)$, and since $f(y)$ is a cluster point of $\{f(y_m)\}_m$, we can find m so that $f(Y \cap V_m) \subseteq Z \setminus f(Y \cap \overline{W}^{cY})$. This leads to a contradiction, however, since $p \in W \cap V_m$, and so $Y \cap W \cap V_m \neq \emptyset$.

Let X be a *hemicompact* space (i.e. in the family of all compact subspaces of X ordered by inclusion, there exists a countable cofinal subfamily [En]). If X is also locally compact, then there exists a sequence $\{C_n\}_n$ of compact sets covering X such that $C_n \subseteq \mathrm{int}C_{n+1}$ (assume $C_0 = \emptyset$). By Proposition 2.2, $\mathcal{P}$ with the Fell topology restricted from $CL(X \times cY)$ coincides with τ_C. Then $\mathcal{H}$, the closure of $\mathcal{P}$ in $(CL(X \times cY), \tau_F)$, is locally compact and hence an open subspace of its Alexandroff one-point compactification $\alpha\mathcal{H}$.

Given a sequence $\{\mathcal{V}_m\}_m$ of cY-open covers of Y, $m, n \in \omega$, a finite (possibly empty) collection $\mathcal{U}$ of nonempty X-open subsets of C_{n+1}, and $\varphi : \mathcal{U} \to \mathcal{V}_m$, the set

$$H_{m,n}(\mathcal{U}, \varphi) = \mathcal{H} \cap (((C_n \setminus \cup\mathcal{U}) \times cY)^c)^+ \cap \bigcap_{U \in \mathcal{U}} ((U \times cY)^- \cap ((\overline{U} \times (cY \setminus \varphi(U)))^c)^+)$$

is open in $\alpha\mathcal{H}$. Let $\mathcal{H}_{m,n}$ be the collection of all these $H_{m,n}(\mathcal{U}, \varphi)$'s. Note that $f \in \mathcal{P} \cap H_{m,n}(\mathcal{U}, \varphi)$ if and only if the following conditions are satisfied (denote $B = dom f$):

(i) $B \cap C_n \subseteq \bigcup \mathcal{U}$,
(ii) $B \cap U \neq \emptyset$ for all $U \in \mathcal{U}$,
(iii) $f(B \cap \overline{U}) \subseteq \varphi(U)$ for all $U \in \mathcal{U}$.

We are now ready to prove some of our main results:

Theorem 3.1. *Let X be locally compact and hemicompact.*

(i) *If Y is Čech-complete with a G_δ-diagonal, then $(\mathcal{P}, \tau_C)$ is Čech-complete.*
(ii) *If Y is a cp-space with a G_δ-diagonal, then $(\mathcal{P}, \tau_C)$ is a p-space.*

Proof. (i) We will show that $\mathcal{P}$ is G_δ in $\alpha\mathcal{H}$: let $\{\mathcal{V}_m\}_m$ be a sequence of covers of Y consisting of cY-open sets that witnesses both Čech-completeness and the G_δ-diagonal property of Y. We claim that

$$(*) \qquad \mathcal{P} = \bigcap_{m,n\in\omega} (\bigcup \mathcal{H}_{m,n}).$$

Indeed, first take $f \in \mathcal{P}$, $m,n \in \omega$, and denote $B = dom f$. If $B \cap C_n = \emptyset$, then $f \in H_{m,n}(\emptyset,\emptyset) \in \mathcal{H}_{m,n}$. If $B \cap C_n \neq \emptyset$, then by continuity of f, and compactness of $B \cap C_n$, there are finite families $\mathcal{U}$ of X-open subsets of C_{n+1} and $\mathcal{W} \subseteq \mathcal{V}_m$ such that $B \cap C_n \subseteq \cup\mathcal{U}$, and for every $U \in \mathcal{U}$, $U \cap B \neq \emptyset$ and there is $W_U \in \mathcal{W}$ with $f(\overline{U} \cap B) \subseteq W_U$. Define $\varphi : \mathcal{U} \to \mathcal{V}_m$ via $\varphi(U) = W_U$ for every $U \in \mathcal{U}$. Then $f \in H_{m,n}(\mathcal{U},\varphi) \in \mathcal{H}_{m,n}$.

Conversely, let $D \in \bigcap_{m,n\in\omega}(\bigcup \mathcal{H}_{m,n})$. We will show that for every $x \in p_X(D)$, $D[x] \subseteq Y$ and $D[x]$ is a singleton, so D is a closed graph of a function with a compact range; thus, $D \in \mathcal{P}$. Indeed, fix $m \in \omega$ and find $n \in \omega$ such that $x \in C_n$. There is a finite family $\mathcal{U}$ of nonempty X-open subsets of C_{n+1}, and $\varphi : \mathcal{U} \to \mathcal{V}_m$ such that $D \in H_{m,n}(\mathcal{U},\varphi)$.

Then there is a $U \in \mathcal{U}$ containing x (otherwise, $H_{m,n}(\mathcal{U},\varphi) \subseteq ((\{x\}\times cY)^c)^+ \cap \mathcal{H}$, and $x \notin p_X(D)$, so $D[x] \subseteq \varphi(U) \subseteq \cup\mathcal{V}_m$; thus, $D[x] \subseteq \bigcap_m(\cup\mathcal{V}_m) = Y$ by Čech-completeness of Y. To prove that $D[x]$ is a singleton, suppose there are distinct $z, y \in D[x]$. Then, by the G_δ-diagonal property, $z \in Y \cap \bigcap_m \mathrm{St}(y,\mathcal{V}_m) = \{y\}$, a contradiction.

(ii) Let $\{\mathcal{V}_m\}_m$ be a sequence of covers of Y consisting of cY-open sets that witness both the *cp*-space and G_δ-diagonal property of Y. We claim that

$$\{\mathcal{H}_{m,n} : m,n \in \omega\}$$

is a feathering for $\mathcal{P}$: to show that each $\mathcal{H}_{m,n}$ is an $\alpha\mathcal{H}$-open cover of $\mathcal{P}$, we can argue as in the first inclusion of $(*)$.

Now choose $f_0 \in \mathcal{P}$ with $B_0 = dom f_0$, and consider $D \in \bigcap_{m,n} \mathrm{St}(f_0,\mathcal{H}_{m,n})$. As in (i), it suffices to show that for every $x \in p_X(D)$, $D[x] \subseteq Y$, and $D[x]$ is a singleton: let $n \in \omega$ be such that $x \in C_n$. For every $m \in \omega$ there is a finite family $\mathcal{U}$ of nonempty X-open subsets of C_{n+1} and a $\varphi : \mathcal{U} \to \mathcal{V}_m$ such that $D, f_0 \in H_{m,n}(\mathcal{U},\varphi)$. Then B_0 intersects every element of $\mathcal{U}$, and there is a $U \in \mathcal{U}$ containing x, so we have

$$D[x] \cup f_0(B_0 \cap \overline{U}) \subseteq \varphi(U) \in \mathcal{V}_m.$$

Consequently, by the *cp*-space property of Y,

$$D[x] \subseteq \bigcap_m \mathrm{St}(f_0(B_0 \cap C_{n+1}),\mathcal{V}_m) \subseteq Y.$$

To prove that $D[x]$ is a singleton, we can use the argument from (i). □

A *sieve* (cf. [Gr]) of Y in cY is a pair (G,T), where $(T,<)$ is an indexing tree of height ω, and G is a decreasing function from T into the nonempty cY-open sets (i.e. $t \leq t'$ implies $G(t) \supseteq G(t')$) such that the sets corresponding to the initial level of T cover Y, and for each $t \in T$,

$$G(t) \cap Y = \bigcup\{G(t') \cap Y : t' \text{ is an immediate successor of } t\}.$$

A *thread* of the sieve (G,T) is the G-image of an infinite branch of T. If Y is a W_δ *subset* of cY - i.e. there is a sieve of Y in cY each thread of which intersects in a (nonempty) subset of Y [CCN] - then Y is called *sieve complete* (the term was first used in [Mi1], but *monotonically Čech complete* [CCN], and satisfying *Condition* $\mathcal{K}$ [WW] was also used; see also [Te]). Note that sieve completeness is independent of the compactification cY [CCN, Proposition 2.11]. It is known that sieve complete spaces are the continuous open images of Čech-complete spaces [WW], so Čech-complete spaces are sieve complete; on the other hand, paracompact sieve complete spaces are Čech-complete [Mi1]. A space Y has a W_δ-*diagonal* [CCN], provided it has a sieve in cY, such that if $\{V_m\}_m$ is any thread of it, and $y \in \bigcap_m V_m \cap Y$, then $\{y\} = \bigcap_m V_m \cap Y$. If Y has a G_δ-diagonal, it has a W_δ-diagonal as well.

Theorem 3.2. *Let X be locally compact and hemicompact. If Y is sieve complete with a W_δ-diagonal, then $(\mathcal{P}, \tau_C)$ is sieve complete.*

Proof. Let (G, T) be a sieve of Y in cY witnessing both sieve completeness and the W_δ-diagonal property of Y. Let T_m stand for the m-th level of T, and denote $\mathcal{V}_m = \{G(t) : t \in T_m\}$ for all m. Inductively define a tree S, and a function H from S to the nonempty $\alpha\mathcal{H}$-open sets as follows: $S = \bigcup_m S_m$ will be a subtree of $\mathcal{P}^{<\omega}$, the tree of finite sequences of points of $\mathcal{P}$, where $S_m \subseteq \mathcal{P}^m$ is the m-th level of S. Let $S_0 = \{\emptyset\}$, and put $H(\emptyset) = H_{0,0}(\emptyset, \emptyset) = \mathcal{H}$.

Let $S_1 = \{(f) : f \in \mathcal{P}\}$. To define $H(f)$ for $f \in \mathcal{P}$, first find the smallest n for which $C_n \cap dom f \neq \emptyset$, and using compactness of $C_n \cap dom f$, get appropriate $\mathcal{U}, \varphi$ such that $f \in H_{1,n}(\mathcal{U}, \varphi)$. Finally, let $H(f)$ be an $\alpha\mathcal{H}$-open set such that

$$f \in H(f) \subseteq \overline{H(f)}^{\alpha\mathcal{H}} \subseteq H_{1,n}(\mathcal{U}, \varphi).$$

Fix $m \geq 1$. Assume that S_m has been defined, and for each $s \in S_m$, $H(s)$ has been chosen so that $\overline{H(s)}^{\alpha\mathcal{H}} \subseteq H(s') \cap H_{m,n_s}(\mathcal{U}_s, \varphi_s)$ for some nonempty $\mathcal{U}_s, \varphi_s$, where $s' \in S_{m-1}$ is the predecessor of s, and $n_s = n_{s'} + 1$. Given $s \in S_m$, $(s, f) \in \mathcal{P}^{m+1}$ will be an immediate successor of s in S, if $f \in \mathcal{P} \cap H(s)$, so

$$S_{m+1} = \{(s, f) : s \in S_m, f \in \mathcal{P} \cap H(s)\}.$$

Let $(s, f) \in S_{m+1}$, and $B = dom f$. Then $f \in H_{m,n_s}(\mathcal{U}_s, \varphi_s)$, so $B \cap C_{n_s} \subseteq \bigcup \mathcal{U}_s$, and for all $U \in \mathcal{U}_s$, $U \subseteq C_{n_s+1}$, $B \cap U \neq \emptyset$, and $f(B \cap \overline{U}) \subseteq \varphi_s(U)$. Fix $U \in \mathcal{U}_s$; then $\varphi_s(U) = G(t)$ for some $t \in T_m$. Consider the collection

$$\mathcal{W}(U) = \{G(t') \in \mathcal{V}_{m+1} : t' \in T_{m+1}, t' > t\},$$

for which $Y \cap \varphi_s(U) = \bigcup\{Y \cap W : W \in \mathcal{W}(U)\}$, and $W \subseteq \varphi_s(U)$ for all $W \in \mathcal{W}(U)$. Since $\bigcup_{U \in \mathcal{U}_s} \mathcal{W}(U)$ is a cY-open cover of $f(B \cap C_{n_s})$, each $x \in B \cap C_{n_s}$ has an X-open neighborhood O_x so that $\overline{O_x} \subseteq U$ for some $U \in \mathcal{U}_s$, and $f(B \cap \overline{O_x}) \subseteq W$ for some $W \in \mathcal{W}(U)$.

By compactness of $B \cap C_{n_s}$, we can find $x_0, \ldots, x_p \in B \cap C_{n_s}$ such that $B \cap C_{n_s} \subseteq \bigcup_{i \leq p} O_{x_i}$, and for all $i \leq p$, there is some $W_i \in \bigcup_{U \in \mathcal{U}_s} \mathcal{W}(U)$ with $f(B \cap \overline{O_{x_i}}) \subseteq W_i$. Denote

$$C = C_{n_s+1} \setminus \bigcup_{i \leq p} O_{x_i},$$

and let $\mathcal{N} = \emptyset$, if $B \cap C = \emptyset$. If $B \cap C \neq \emptyset$ then, because $\mathcal{V}_{m+1}$ covers Y, each $z \in B \cap C$ has an X-open neighborhood N_z such that $\overline{N_z} \subseteq \mathrm{int} C_{n_s+2} \setminus C_{n_s}$, and $f(B \cap \overline{N_z}) \subseteq V$ for some $V \in \mathcal{V}_{m+1}$. By compactness of $B \cap C$, we can find $z_0, \ldots, z_r \in B \cap C$ such that $B \cap C \subseteq \bigcup_{j \leq r} N_j$, and for all $j \leq r$ there is some $V_j \in \mathcal{V}_{m+1}$ with $f(B \cap \overline{N_{z_j}}) \subseteq V_j$; put $\mathcal{N} = \{N_{z_j} : j \leq r\}$.

Define $\mathcal{U}_{(s,f)} = \{O_{x_i} : i \leq p\} \cup \mathcal{N}$, and

$$\varphi_{(s,f)}(A) = \begin{cases} W_i, & \text{if } A = O_{x_i} \text{ for some } i \leq p, \\ V_j, & \text{if } A = N_{z_j} \text{ for some } j \leq r. \end{cases}$$

It is clear from the above construction, that $f \in H_{m+1,n_s+1}(\mathcal{U}_{(s,f)}, \varphi_{(s,f)})$, so we can find an $\alpha\mathcal{H}$-open set $H(s,f)$ such that putting $n_{(s,f)} = n_s + 1$ we have

$$f \in H(s,f) \subseteq \overline{H(s,f)}^{\alpha\mathcal{H}} \subseteq H(s) \cap H_{m+1,n_{(s,f)}}(\mathcal{U}_{(s,f)}, \varphi_{(s,f)}).$$

Then $H(s,f) \subseteq H(s)$, and $\mathcal{P} \cap H(s) = \bigcup\{\mathcal{P} \cap H(s,f) : f \in \mathcal{P} \cap H(s)\}$, so (H,S) is a sieve of $\mathcal{P}$ in $\alpha\mathcal{H}$.

Finally, let $\{H(s_m) : m \geq 1\}$ be a thread in (H,S), where for all m, $s_m \in S_m$, and s_{m+1} is an immediate successor of s_m in S. Let $\mathcal{U}_{s_m}$ be a finite collection of X-open sets, and $\varphi_{s_m} : \mathcal{U}_{s_m} \to \mathcal{V}_m$ such that

$$\overline{H(s_m)}^{\alpha\mathcal{H}} \subseteq H(s_{m-1}) \cap H_{m,n_{s_m}}(\mathcal{U}_{s_m}, \varphi_{s_m}).$$

Since $\alpha\mathcal{H}$ is compact, then $\bigcap_m H(s_m) \neq \emptyset$, so we can choose $D \in \bigcap_m H(s_m)$, and $x \in p_X(D)$. Find the first m with $x \in C_{n_{s_m}}$. Then the set

$$T'_k = \{t \in T_k : \exists U \in \mathcal{U}_{s_k} \text{ with } x \in U \text{ and } G(t) = \varphi_{s_k}(U)\}$$

is nonempty for each $k \geq m$. Moreover, if $t \in T'_k$ for some $k > m$, and $U \in \mathcal{U}_{s_k}$ is such that $G(t) = \varphi_{s_k}(U)$, then by the definition of $\mathcal{U}_{s_k}$, either U is disjoint from $C_{s_{k-1}}$, which is not the case, since $x \in C_{s_m} \cap U \subseteq C_{s_{k-1}} \cap U$, or there exists a $U' \in \mathcal{U}_{s_{k-1}}$ with $U \subseteq U'$ and $G(t) \in \mathcal{W}(U')$. Then there is $t' \in T_{k-1}$ with $t' < t$, and $G(t') = \varphi_{s_{k-1}}(U')$; thus, $t' \in T'_{k-1}$. It means that $T' = \bigcup_{k \geq m} T'_k$ is a subtree of T of height ω, which is clearly finite splitting (i.e. each node in T' has only finitely many immediate successors - see [Ke]). By König's Lemma, T' has an infinite branch

$$t_m < \cdots < t_k < t_{k+1} < \ldots$$

where $t_k \in T'_k$ for each $k \geq m$. Let $U_k \in \mathcal{U}_{s_k}$ be such that $\varphi_{s_k}(U_k) = G(t_k)$. Then $x \in U_k$, and $D[x] \subseteq \varphi_{s_k}(U_k)$ for all $k \geq m$, so $\{\varphi_{s_k}(U_k) : k \geq m\}$ is part of a thread of (G,T) containing $D(x)$. This thread is intersecting in a singleton $y \in Y$, since Y is sieve complete with a W_δ-diagonal. It follows, that $y = D(x)$; thus, $D \in \mathcal{P}$. □

A space Y is a *q-space* [Mi2, Ch], if for each $y \in Y$ there is a sequence $\{U_n\}_n$ of neighborhoods of y such that whenever $y_n \in U_n$, the sequence $\{y_n\}_n$ has a cluster point. Note that sieve complete, as well as, p-spaces are q-spaces.

Proposition 3.3. *Let Y be a non-countably compact space. If $(\mathcal{P}, \tau_C)$ is a q-space, then X is hemicompact.*

Proof. Let $y \in Y$, and define $f(x) = y$ for each $x \in X$. Let $\mathcal{U}_n = \bigcap_{p \in P_n}[U_n^p] \cap \bigcap_{r \in R_n}[K_n^r : I_n^r]$ (where P_n, R_n are finite sets) be a sequence of τ_C-neighborhoods of f satisfying the q-space property at f.

For every $p \in P_n$ choose $x_n^p \in U_n^p$ and put $K_n = \{x_n^p : p \in P_n\} \cup \bigcup_{r \in R_n} K_n^r$. Then K_n is compact for every n, further, $\{K_n\}_n$ is a countable cofinal subfamily of $K(X)$: otherwise, let $K \in K(X)$ be such that for every $n \in \omega$ there is $k_n \in K \setminus K_n$. Let $\{y_n\}_n$ be a sequence without a cluster point in Y. Define $f_n : K_n \cup \{k_n\} \to Y$ via $f_n(x) = f(x)$ for $x \in K_n$, and $f_n(k_n) = y_n$. It follows, that the sequence $f_n \in \mathcal{U}_n$ clusters in some $h \in \mathcal{P}$. The set $h(K \cap \mathit{dom}\, h)$ is compact, so there is an open $I \subseteq Y$ and $n_0 \in \omega$ such that $h(K \cap dom\, h) \subseteq I$, and $y_n \notin I$ for every $n \geq n_0$. Then $[K : I]$ is a τ_C-neighborhood of h, and for every $n \geq n_0$, $f_n \notin [K : I]$, which is a contradiction. $\square$

Remark 3.4. If Y is countably compact, the above proposition may fail: indeed, if $Y = \{0, 1\}$ and X is an uncountable discrete space, then $(\mathcal{P}, \tau_C)$ is a closed subspace of $(CL(X \times Y), \tau_F)$, so it is Čech-complete, but X is not hemicompact. $\square$

Theorem 3.5. *Let Y be a non-countably compact space with a G_δ-diagonal (W_δ-diagonal). Then the following are equivalent:*

(i) *$(\mathcal{P}, \tau_C)$ is Čech-complete (sieve complete);*
(ii) *X is locally compact, Lindelöf, and Y is Čech-complete (sieve complete).*

Proof. (i)⇒(ii) By Proposition 2.4, X is locally compact, by Proposition 3.3, X is hemicompact, and hence Lindelöf. Moreover, Y is Čech-complete (sieve complete) by Proposition 2.1(ii).

(ii)⇒(i) See Theorem 3.1(i), and Theorem 3.2. $\square$

Theorem 3.6. *Let Y be a non-compact, paracompact space with a G_δ-diagonal. Then the following are equivalent:*

(i) *$(\mathcal{P}, \tau_C)$ is a p-space;*
(ii) *X is locally compact, Lindelöf, and Y is metrizable.*

Proof. (i)⇒(ii) By Proposition 2.4, X is locally compact, by Proposition 3.3, X is hemicompact, and hence Lindelöf. Moreover, Y is a p-space by Proposition 2.1(ii), and a paracompact p-space with a G_δ-diagonal is metrizable [Gr].

(ii)⇒(i) See Theorem 3.1(ii). $\square$

The following theorem is an extension of [Ho2, Theorem 3.3] (for another proof see [NZ]):

Theorem 3.7. *The following are equivalent:*

(i) *$(\mathcal{P}, \tau_C)$ is completely metrizable;*
(ii) *X is hemicompact, metrizable, and Y is completely metrizable.*

Proof. (i)⇒(ii) Y is completely metrizable by Proposition 2.1(ii); further, metrizability of $(\mathcal{P}, \tau_C)$ implies metrizability of $(CL(X), \tau_F)$ (see Proposition 2.1(i)), which in turn is equivalent to hemicompactness and metrizability of X [Be, Theorem 5.1.5].

(ii)⇒(i) $(\mathcal{P}, \tau_C)$ is metrizable by [Ho2, Theorem 2.4], and Čech-complete by our Theorem 3.1, hence it is completely metrizable [En, Theorem 4.3.26]. $\square$

Finally, using the argument of [Ho2, Theorem 3.4], we can fully characterize Polishness of $(\mathcal{P}, \tau_C)$:

Theorem 3.8. *The following are equivalent:*

(i) *$(\mathcal{P}, \tau_C)$ is Polish;*
(ii) *X is hemicompact, metrizable, and Y is Polish.*

4. Čech-completeness and related properties of the compact-open topology on $C(X,Y)$

Properties of the compact-open topology τ_{CO} are well established for $Y=\mathbb{R}$, or more generally for a locally convex completely metrizable Y (cf. [MN1, En], or more recently [GRe]). This restriction on Y is usually required to use some extension theorem (Tietze, Dugundji) in order to obtain the desired result. As we have shown in previous sections, one can get by in $(\mathcal{P},\tau_C)$ without extension theorems, and with considerably more general Y. On the other side, $(C(X,Y),\tau_{CO})$ embeds as a closed subset in $(\mathcal{P},\tau_C)$, which leads to results on τ_{CO} with less restricted Y. For theorems of this general nature see e.g. [Are], [Mi3], [OM] or [SM].

To be more specific, note that if X is a hemicompact k-space and Y *is completely metrizable*, then $(C(X,Y),\tau_{CO})$ is completely metrizable [MN1], and hence Čech-complete, sieve complete, as well as a p-space. As a corollary to Theorem 3.2 (resp. Theorem 3.1) , we can generalize this result to a *sieve complete (resp. Čech-complete, cp-space) Y with a W_δ-diagonal (resp. G_δ-diagonal).* For another proof see [HZ2]:

Theorem 4.1. *Let X be a hemicompact k-space, and Y be sieve complete (Čech-complete, cp-space, resp.) with a W_δ-diagonal (G_δ-diagonal, resp.).*

Then $(C(X,Y),\tau_{CO})$ is sieve complete (Čech-complete, p-space, resp.).

Proof. Let $\{C_n\}_n$ be a cofinal collection in $K(X)$, and $Z=\bigoplus_n C_n$ the topological sum of the C_n's. Then Z is hemicompact, locally compact and, since X is a k-space, the natural mapping $\psi : Z\to X$ is compact-covering and quotient. Consequently, the map $\psi^* : (C(X,Y),\tau_{CO})\to(C(Z,Y),\tau_{CO})$, defined via

$$\psi^*(f)=f\circ\psi, \text{ for all } f\in C(X,Y),$$

is a closed embedding [MN1, Corollary 2.2.8(b), and Theorem 2.2.10]. It follows from Theorem 3.2 (Theorem 3.1, resp.), and Proposition 2.1(ii), that $(C(Z,Y),\tau_{CO})$ is sieve complete (Čech-complete, p-space, resp.), as is $(C(X,Y),\tau_{CO})$, which sits in $(C(Z,Y),\tau_{CO})$ as a closed subspace. □

5. Paracompactness and related properties of $(\mathcal{P},\tau_C)$

Since a paracompact p-space with a G_δ-diagonal is metrizable [Gr], we have

Lemma 5.1. *A locally compact, Lindelöf space with a G_δ-diagonal is metrizable.*

Theorem 5.2. *Let Y be a Polish space. The following are equivalent:*

(i) *$(\mathcal{P},\tau_C)$ is Polish;*
(ii) *$(\mathcal{P},\tau_C)$ is Lindelöf;*
(iii) *X is hemicompact, and metrizable.*

Proof. For (iii)⇔(i) see Theorem 3.8. As for (ii)⇒(iii), observe that $(\mathcal{P},\tau_C)$ Hausdorff implies X is locally compact (Proposition 2.4), so by Proposition 2.1(i), $(CL(X),\tau_F)$ is Lindelöf, and so is X. This in turn implies that $(C(X,Y),\tau_{CO})$ is metrizable [MN1, Exercise 4.9.1] and, by Proposition 2.1(ii), Lindelöf; thus, 2nd countable. Then, by [MN1, Exercise 4.9.7(a)], X is an $\aleph_0$-space (i.e. it has a countable k-network [Gr]). Finally, since an $\aleph_0$-space has a G_δ-diagonal, X is metrizable by Lemma 5.1. □

Proposition 5.3. *Let Y be sieve complete (Čech-complete, cp-space, resp.) with a W_δ-diagonal (G_δ-diagonal, resp.). If $(\mathcal{P},\tau_C)$ is normal, then $(\mathcal{P},\tau_C)$ is sieve complete (Čech-complete, a p-space, resp.).*

Proof. $(CL(X),\tau_F)$ is normal since, by Proposition 2.4 and Proposition 2.1(i), it embeds as a closed subset in $(\mathcal{P},\tau_C)$, so X is locally compact and hemicompact [HLP]. By Theorem 3.2 (Theorem 3.1), $(\mathcal{P},\tau_C)$ is sieve complete (Čech-complete, a p-space, resp.). □

If we restrict our attention to spaces with G_δ-diagonals, it is possible to extend Theorem 5.2:

Theorem 5.4. *Let X have a G_δ-diagonal, and Y be a non-compact Polish space. Then the following are equivalent:*

(i) *$(\mathcal{P},\tau_C)$ is Polish;*
(ii) *$(\mathcal{P},\tau_C)$ is normal;*
(iii) *X is locally compact, and Lindelöf.*

Proof. (iii)⇒(i) follows from Lemma 5.1, and Theorem 3.8.
(ii)⇒(iii) follows from Proposition 5.3, and Theorem 3.5. □

Proposition 5.5. *$(\mathcal{P},\tau_C)$ is paracompact if and only if it is Lindelöf, if either of the following is satisfied:*

(i) *X has a G_δ-diagonal and Y is 2nd countable;*
(ii) *X, Y are separable.*

Proof. The nontrivial part is to prove that paracompactness implies Lindelöfness. Paracompactness of $(\mathcal{P},\tau_C)$ implies paracompactness of $(CL(X),\tau_F)$ (by Proposition 2.4 and Proposition 2.1(i)), so X is locally compact and Lindelöf [HLP].

Then by Lemma 5.1, (i) implies (ii), so it suffices to prove Lindelöfness of $(\mathcal{P},\tau_C)$ for (ii): let D_X, D_Y be countable dense sets in X and Y, respectively. We will be done if we show that $(\mathcal{P},\tau_C)$ is separable: by [HZ1, Proposition 2.1(iii)], the collection of sets

$$[K_0 : \emptyset] \cap \bigcap_{i\le n} [U_i] \cap [\overline{U_i} : I_i],$$

with $n \ge 1, K_0, \overline{U_i} \in K(X)$, $\emptyset \ne U_i \subseteq X$ open, K_0, U_i pairwise disjoint for $i \le n$ and $\emptyset \ne I_i \subseteq Y$ open ($i \le n$), forms a π-base for τ_C. Using this π-base, it is not hard to verify that the collection of continuous partial maps with values in D_Y, the domains of which run over the finite subsets of D_X, is dense (and countable) in $(\mathcal{P},\tau_C)$. □

Remark 5.6. Note that paracompactness of $(\mathcal{P},\tau_C)$ is not equivalent to Lindelöfness and local compactness of X (as Theorem 5.4 would suggest), since otherwise (comparing Theorem 5.2 and Proposition 5.5(ii)), a separable, locally compact, Lindelöf space would be metrizable (which is not the case, just consider $\beta\omega$). The following results suggest rather, that paracompactness of $(\mathcal{P},\tau_C)$ could be closer to metrizability of $(\mathcal{P},\tau_C)$. □

Proposition 5.7. *Let Y be a p-space with a G_δ-diagonal. The following are equivalent:*

(i) $(\mathcal{P}, \tau_C)$ *is paracompact;*
(ii) $(\mathcal{P}, \tau_C)$ *is a paracompact p-space;*

Proof. (i)⇒(ii) follows from Proposition 5.3, since Y is paracompact by Proposition 2.1(ii), and a paracompact p-space is a cp-space. □

Theorem 5.8. *Let Y be a p-space. The following are equivalent:*

(i) $(\mathcal{P}, \tau_C)$ *is paracompact with a G_δ-diagonal;*
(ii) $(\mathcal{P}, \tau_C)$ *is metrizable;*
(iii) X *is hemicompact metrizable, and Y is metrizable.*

Proof. (i)⇒(ii) It follows from Proposition 2.1(ii), that Y has a G_δ-diagonal, so by Proposition 5.7, $(\mathcal{P}, \tau_C)$ is a paracompact p-space with a G_δ-diagonal, and hence, metrizable [Gr]. The remaining implications follow from [Ho2, Theorem 2.4]. □

The following lemma is needed to prove our last characterization of paracompactness of $(\mathcal{P}, \tau_C)$, but it may be of independent interest:

Lemma 5.9. *Let X, Y be such that partial continuous functions with closed domains are continuously extendable over X; moreover, suppose that there exists an open base $\mathcal{V}$ for Y closed under finite intersections such that for each nonempty $K \in K(X)$ and $V \in \mathcal{V}$, every function $f \in C(K, V)$ is extendable to some $f^* \in C(X, V)$.*

Then the set-valued function $\psi : (\mathcal{P}, \tau_C) \rightrightarrows (C(X, Y), \tau_{CO})$ defined via

$$\psi(f) = \{g \in C(X, Y) : g \restriction_{dom f} = f\}$$

is lower semicontinuous.

Proof. The restriction mapping

$$\eta : (CL(X), \tau_F) \times (C(X, Y), \tau_{CO}) \to (\mathcal{P}, \tau_C),$$

defined via $\eta(B, f) = f \restriction_B$, is an open mapping (see [HZ1, Proposition 3.3]). It suffices to notice that $\psi^{-1}(\mathbf{V}) = \eta(CL(X) \times \mathbf{V}) \in \tau_C$ for any $\mathbf{V} \in \tau_{CO}$. □

Theorem 5.10. *Let Y be a Frèchet space. The following are equivalent:*

(i) $(\mathcal{P}, \tau_C)$ *is paracompact;*
(ii) X *is locally compact, Lindelöf, and there exists a continuous mapping $\varphi : (\mathcal{P}, \tau_C) \to (C(X, Y), \tau_{CO})$ such that $\varphi(f)$ is a continuous extension of $f \in \mathcal{P}$;*
(iii) X *is locally compact, Lindelöf, and $(\mathcal{P}, \tau_C)$ embeds as a closed subset in $(CL(X), \tau_F) \times (C(X, Y), \tau_{CO})$.*

Proof. (i)⇒(ii) X is locally compact and Lindelöf by [HLP], since $(CL(X), \tau_F)$ is paracompact by Proposition 2.4, and Proposition 2.1(i). Now use Lemma 5.9 along with Michael's Selection Theorem, as in the proof of [Ho2, Theorem 4.1], to finish the proof.

(ii)⇒(iii) See the proof of [Ho2, Theorem 4.2].

(iii)⇒(i) If X is locally compact and Lindelöf, then $(CL(X), \tau_F)$ is locally compact, paracompact [HLP] and $(C(X, Y), \tau_{CO})$ is metrizable [MN1]; hence, the product $(CL(X), \tau_F) \times (C(X, Y), \tau_{CO})$ is paracompact [En, Exercise 5.5.5.(c)]. □

Paracompactness and normality have been studied for the Fell topology [HLP], as well as for the compact-open topology [Po],[OM]. For the generalized compact-open topology we only have the above partial results, so we ask:

Problem 5.11. *Characterize paracompactness (normality) of $(\mathcal{P}, \tau_C)$ in terms of X and Y. In particular, is paracompactness (normality) of $(\mathcal{P}, \tau_C)$ equivalent to its metrizability for some reasonable Y (say $Y = \mathbb{R}$)?*

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References

[AA] A. Abd-Allah, *Partial maps in algebra and topology*, Ph.D. thesis, University of Wales, 1979.

[AB] A. Abd-Allah and R. Brown, *A compact-open topology on partial maps with open domains*, J. London Math. Soc. **21** (1980), 480-486.

[Are] R. Arens, *A topology of transformations*, Annals of Math. **47** (1946), 480-495.

[Arh] A. V. Arhangel'skiĭ*A class of spaces containing all metric and locally compact spaces*, Mat. Sb. **67** (1970), 55-58.

[Ba] J. Back, *Concepts of similarity for utility functions*, J. of Math. Economics **1** (1986), 721-727.

[BB] P. I. Booth and R. Brown, *Spaces of partial maps, fibered mapping spaces and the compact open topology*, Topology Appl. **8** (1978), 181-195.

[BC] P. Brandi and R. Ceppitelli, *Existence, uniqueness and continuous dependence for hereditary differential equations*, J. Diff. Equations **81** (1989), 317-339.

[BCH1] P. Brandi, R. Ceppitelli and Ľ. Holá, *Topological properties of a new graph topology*, J. Convex Anal. **5** (1998), 1-12.

[BCH2] P. Brandi, R. Ceppitelli and Ľ. Holá, *Kuratowski convergence on compacta and Hausdorff metric convergence on compacta*, Comment. Math. Univ. Carolin. **40** (1999), 309-318.

[Be] G. Beer, *Topologies on Closed and Closed Convex Sets*, Kluwer, Dordrecht, 1993.

[CCN] J. Chaber, M. M. Čoban and K. Nagami, *On monotonic generalizations of Moore spaces, Čech complete spaces and p-spaces*, Fund. Math. **84** (1974), 107-119.

[Ch] T. Chiba, *On q-spaces*, Proc. Japan Acad. **45** (1969), 453-456.

[DN1] A. Di Concilio and S. A. Naimpally, *Proximal set-open topologies and partial maps*, Acta Math. Hung. **88** (2000), 227-237.

[DN2] A. Di Concilio and S. A. Naimpally, *Function space topologies on (partial) maps*, Recent Progress in Function Spaces, Quaderni di Matematica **3**, 1998.

[En] R. Engelking, *General Topology*, Helderman, Berlin, 1989.

[Fe] J. Fell, *A Hausdorff topology for the closed subsets of locally compact non-Hausdorff space*, Proc. Amer. Math. Soc. **13** (1962), 472–476.

[Fi] V. V. Filippov, *Basic topological structures of the theory of ordinary differential equations*, Topology in Nonlinear Analysis, Banach Centrum Publications **35** (1996), 171-192.

[Gr] G. Gruenhage, *Generalized metric spaces*, in Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984.

[GRe] P. M. Gartside and E. A. Reznichenko, *Near metric properties of function spaces*, Fund. Math. **164** (2000), 97-114.

[HLP] Ľ. Holá, S. Levi and J. Pelant, *Normality and paracompactness of the Fell topology*, Proc. Amer. Math. Soc. **127** (1999), 2193-2197.

[HM] D. Holý and L. Matejička, *C-upper semicontinuous and C^*- upper semicontinuous multifunctions*, Tatra Mt. Math. Publ. **34** (2006), 159-165.

[Ho1] Ľ. Holá, *Uniformizability of the generalized compact-open topology*, Tatra Mt. Math. Publ. **14** (1998), 219-224.

[Ho2] Ľ. Holá, *Complete metrizability of generalized compact-open topology*, Topology Appl. **91** (1999), 159-167.

[HZ1] Ľ. Holá and L. Zsilinszky, *Completeness properties of the generalized compact-open topology on partial functions with closed domains*, Topology Appl. **110** (2001), 303-321.

[HZ2] Ľ. Holá and L. Zsilinszky, *Vietoris topology on partial maps with compact domains*, Topology Appl., to appear.

[Ke] A. S. Kechris *Classical Descriptive Set Theory*, Springer, New York, 1994.

[KS] H. P. Künzi and L. B. Shapiro, *On simultaneous extension of continuous partial functions*, Proc. Amer. Math. Soc. **125** (1997), 1853-1859.

[KT] E. Klein and A. Thompson, *Theory of Correspondences*, Wiley, New York, 1975.

[Ku1] K. Kuratowski, *Sur l'espace des fonctions partielles*, Ann. Mat. Pura Appl. **40** (1955), 61-67.

[Ku2] K. Kuratowski, *Sur une méthode de métrisation complète de certains espaces d'ensembles compacts*, Fund. Math. **43** (1956), 114-138.

[La] H.J. Langen, *Convergence of dynamic programming models*, Mathematics of Operations Research **6** (1981), 493-512.

[Mi1] E. Michael, *Complete spaces and tri-quotient maps*, Illinois J. Math. **21** (1977), 716-733.

[Mi2] E. Michael, *A note on closed maps and compact sets*, Israel J. Math. **2** (1964), 173-176.

[Mi3] E. Michael, $\aleph_0$*-spaces*, J. Math. Mech. **15** (1966), 983-1002.

[MN1] R. A. McCoy and I. Ntantu, *Topological properties of spaces of continuous functions*, Springer-Verlag, Berlin, 1988.

[MN2] R. A. McCoy and I. Ntantu, *Completeness properties of function spaces*, Topology Appl. **22** (1986), 191-206.

[NZ] P. J. Nyikos and L. Zsilinszky, *Strong α-favorability of the (generalized) compact-open topology*, Atti Sem. Mat. Fis. Univ. Modena **51** (2003), 1-8.

[OM] P. O'Meara, *On paracompactness in function spaces with the compact-open topology*, Proc. Am. Math. Soc.**29** (1971), 183-189.

[Po] R. Pol, *Normality in function spaces*, Fund. Math. **84** (1974), 145-155.

[Se] G. R. Sell, *On the Fundamental Theory of Ordinary Differential Equations*, J. Differential Equations **1** (1965), 371-392.

[SM] N. Shimane and T. Mizokami, *On the embedding and developability of mapping spaces with compact-open topology*, Topology Proc. **24** (1999), 313-322.

[St] E. N. Stepanova, *Extension of continuous functions and metrizability of paracompact p-spaces*, Mat. Zametki **53** (1993), 92-101.

[Te] R. Telgársky, *On sieve-complete and compact-like spaces*, Topology Appl. **16** (1983), 61-68.

[Wh] W. Whitt, *Continuity of Markov Processes and Dynamic Programs*, Yale University, 1975.

[WW] H. H. Wicke and J. M. Worrell, Jr., *On the open continuous images of paracompact Čech-complete spaces*, Pacific J. Math. **37** (1971), 265-276.

[Za] S. K. Zaremba, *Sur certaines familles de courbes en relation avec la theorie des equations differentielles*, Rocznik Polskiego Tow. Matemat. **15** (1936), 83-105.

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